Latent voter model on random regular graphs

Shirshendu Chatterjee

Cornell University (visiting Duke U.)

Work in progress with Rick Durrett

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Outline

- Definition of voter model and duality with coalescing random walks
- Voter models on “Complex Networks" (Facebook)
- Definition of Latent voter model (iPad) and mean field equations
- Approximate duality with branching coalescing random walk
- Limiting behavior

Voter model on a graph

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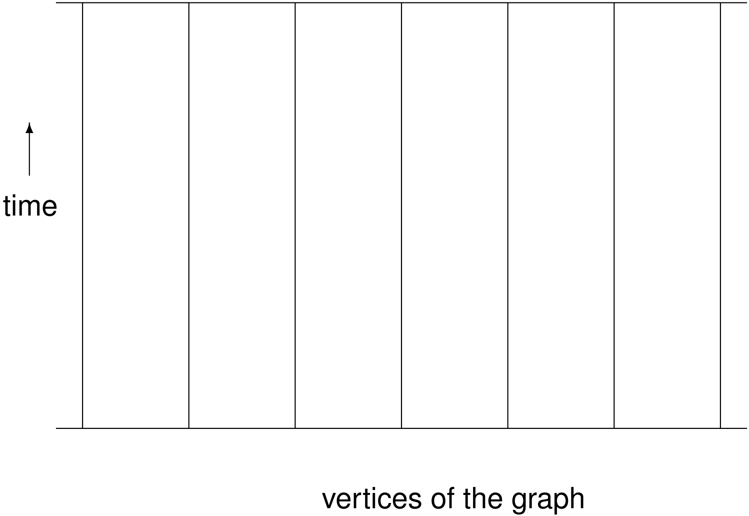
- For a convenient construction of the process we draw an arrow from (x, T_n^x) to (y_n^x, T_n^x).
- Arrows point to the opposite direction of the flow of information. But this choice will be useful in defining a dual process.

Graphical Representation



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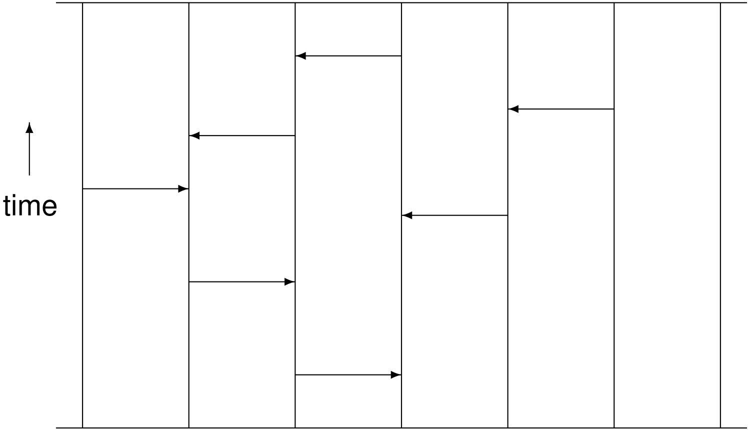
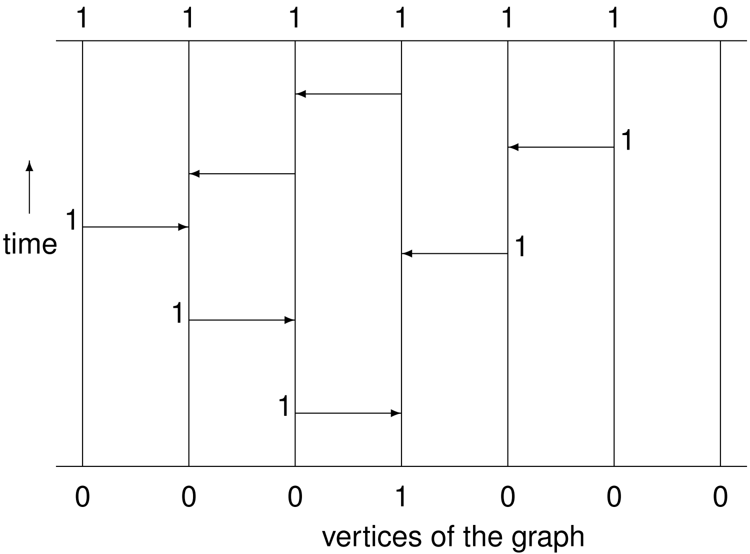


vertices of the graph

Graphical Representation



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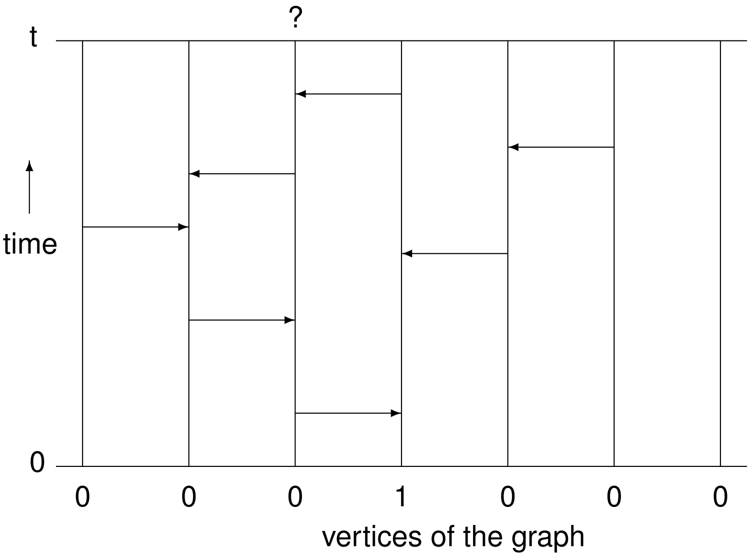

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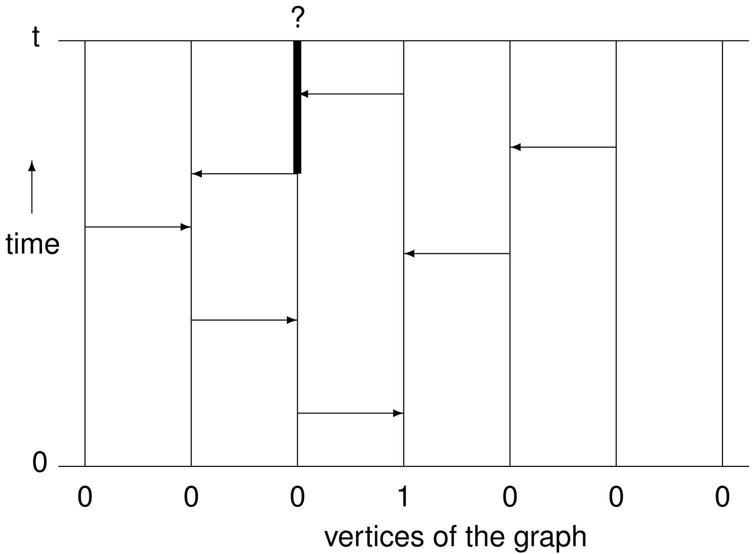

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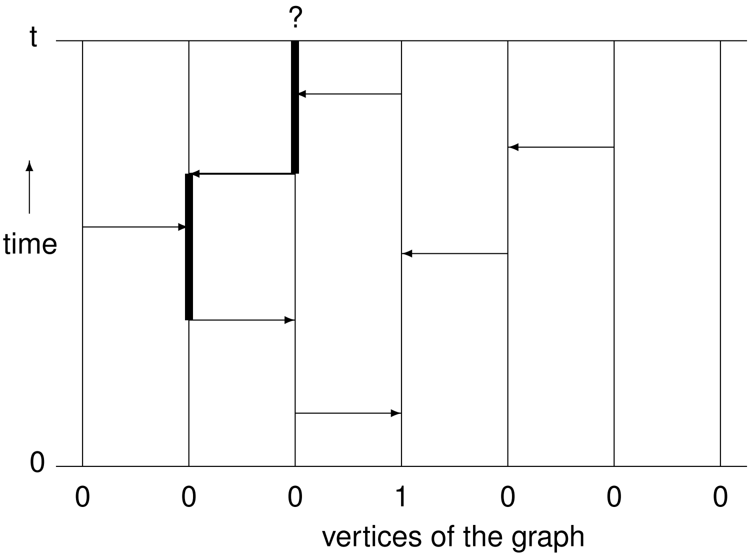

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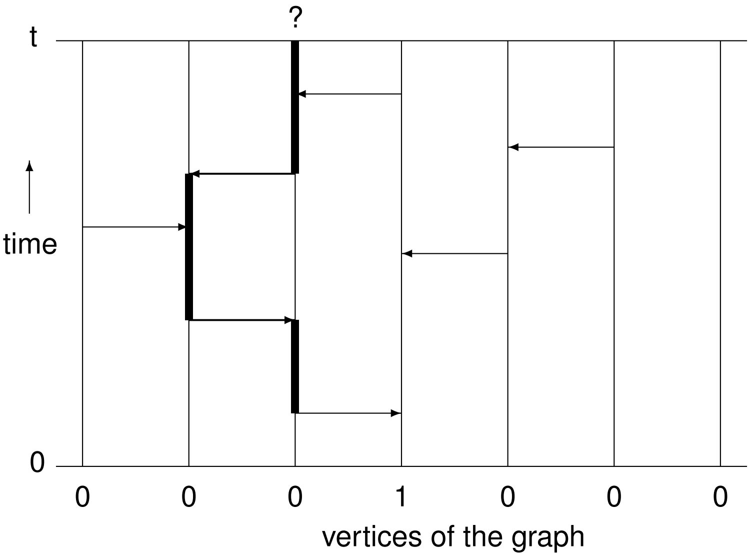

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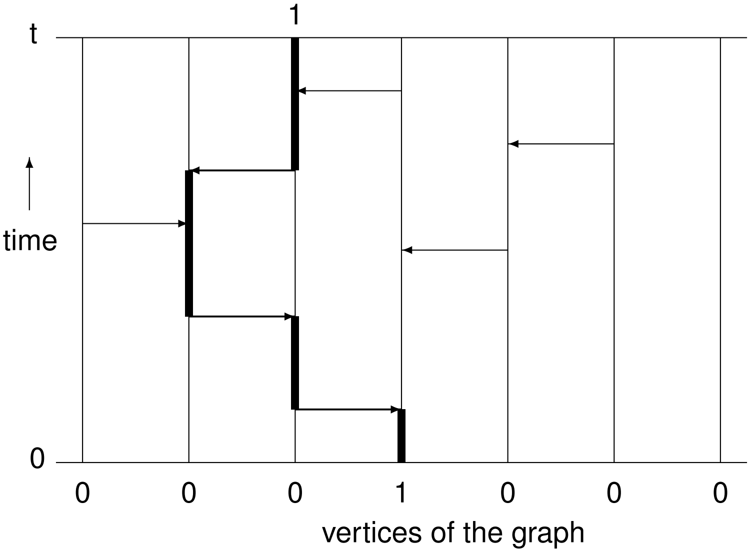
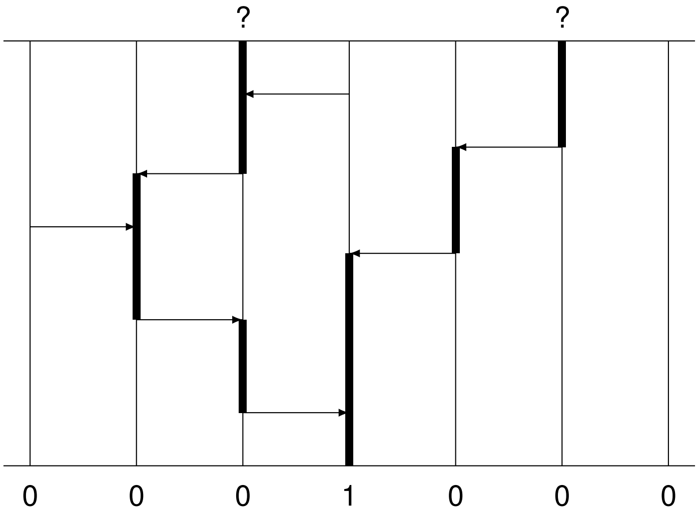


What is x at time t?

- To compute the state of x at time t we work backwards in time to define $\zeta_s^{x,t}$, $s \leq t$.
- $\zeta_r^{x,t} = x$ for $r < s$, the first time so that $t - s = T_n^x$ for some n.
- Then set $\zeta_s^{x,t} = y_n^x$, and repeat.
- For all $0 \leq s \leq t$, $\xi_t(x) = \xi_{t-s}(\zeta_s^{x,t})$.
- Follow the arrow. Jump to the neighbor you imitated.
- $\zeta_s^{x,t}$ is a random walk that jumps $x \to y$ at rate $1/d(x)$ if y is a neighbor of x.

Coalescing random walks



Duality

- $\zeta_s^{B,t} = \{\zeta_s^{x,t} : x \in B\}$ for $s \le t$. (Dual rw's)
- Let $\eta_t^A = \{x : \xi_t(x) = 1\}$ when $A = \{x : \xi_0 = 1\}$.
  Voter model written as a set-valued process.

$$\{\eta_t^A \cap B \neq \emptyset\} = \{\zeta_t^{B,t} \cap A \neq \emptyset\}$$

- Define ζ_s^B, $s \ge 0$ so that $\zeta_s^B =_d \zeta_s^{B,t}$ for $s \le t$

$$P(\eta_t^A \cap B \neq \emptyset) = P(\zeta_t^B \cap A \neq \emptyset)$$

- ζ_t^B is a coalescing random walk: particles move independent until they hit, and coalesce to one when they hit.

Finite graphs G

- The voter model on a finite graph is a finite Markov chain with two absorbing states (all 0's and all 1's) and so eventually it reaches complete consensus.
- Q. How long does it take?
- Let X_t^1 and X_t^2 be independent random walks on the graph.
- Let $A = \{(x, x) : x$ is a vertex of the graph$\}$, $T_A = \inf\{t : X_t^1 = X_t^2\}$
- How big is T_A when X_0^1 and X_0^2 are randomly chosen?
- This is a lower bound on the time to consensus and is the right order of magnitude.

- Write P_π for the law of (X_t^1, X_t^2), $t \geq 0$ when X_0^1 and X_0^2 are independent with distribution π.

- Proposition 23 in Chapter 3 Aldous and Fill's book on *Reversible Markov Chains.*

$$\sup_t |P_\pi(T_A > t) - \exp(-t/E_\pi T_A)| \leq \frac{\tau_2}{E_\pi T_A}$$

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- Why? Make the stronger assumption that the mixing time $t_n \ll E_\pi T_A$. In the limit we have the lack of memory property.

$$P(T_A > (t+s)E_\pi T_A | T_A > sE_\pi T_A) \approx P(T_A > tE_\pi T_A)$$

Computing $E_\pi(T_A)$

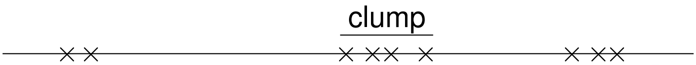
- For the underlying discrete chain, where at each step one particle is chosen at random and allowed to jump,

$$\begin{aligned} 1/\pi(A) &= E_A(T_A) \\ &= o(n) + E_A(T_A | T_A \gg t_n) P_A(T_A \gg t_n) \\ &= o(n) + E_\pi(T_A) P_A(T_A \gg t_n), \end{aligned}$$

  which implies

$$E_\pi(T_A) \approx \frac{1}{\pi(A)} \cdot \frac{1}{P_A(T_A \gg t_n)}.$$

- Naive guess for waiting time must be corrected by multiplying by the expected clump size. Have a geometric number of quick returns with “success probability" $P_A(T_A \gg t_n)$.



Random Regular Graphs

- Sood and Redner (2005) have considered graphs with a power-law degree distribution. Other locally tree-like graphs have same qualitative behavior..
- Rancom regular graph is locally a tree with degree r.
- Distance between two random walks, when positive, increases by 1 with probability $(r-1)/r$ and decreases by 1 with prob. $1/r$

$$P_A(T_A \gg t_n) = 1 - 1/(r-1) = (r-2)/(r-1) \quad \text{for } r \geq 3.$$

If each particle jump at rate 1, then

$$E_\pi(T_A) \approx \frac{1}{2}\frac{1}{\pi(A)} \cdot \frac{1}{P_A(T_A \gg t_n)} = \frac{n(r-1)}{r-2} \cdot \frac{1}{2}$$

Summary for voter model on finite graphs

- Eventually it reaches complete consensus.
- The consensus time for locally-tree like random graphs on n vertices having degree distribution with finite second moment is $O(n)$.
- Sood and Redner (2005) have considered graphs with a power-law degree distribution. In all cases, the consensus time is at most linear in the number of vertices.
- Starting from product measure, the quasi-stationary density of state 0 is the same as the initial proportion.

Latent Voter Model: Motivation

- Lambiottei, Saramaki, and Blondel (2009) Phys. Rev. E. 79, paper 046107
- Consider the states of the voter model to be $0 =$ IBM laptop and $1 =$ iPad. or Blu-Ray versus HD-DVD.
- “It is likely that choice of a customer is influenced by his acquaintances. However it is unlikely that the customer will replace his equipment immediately after a purchase."
- To reflect this, after an opinion change the voter enters an inactive state that lasts for an exponentially distributed amount of time with mean $1/\lambda$.

Mean field equations.

- We write $+$ and $-$ instead of 1 and 0 (or $\uparrow$ and $\downarrow$ in LSB):

$$\frac{d\rho_+}{dt} = \rho_{a-}\rho_+ - \rho_{a+}(1-\rho_+)$$
$$\frac{d\rho_{a+}}{dt} = -\rho_{a+}(1-\rho_+) + \lambda(\rho_+ - \rho_{a+})$$
$$\frac{d\rho_{a-}}{dt} = -\rho_{a-}\rho_+ + \lambda(1-\rho_+ - \rho_{a-})$$

From the second equation we see that in equilibrium

$$0 = -\rho_{a+} + \rho_{a+}\rho_+ + \lambda\rho_+ - \lambda\rho_{a+}$$

Using these in first equation we find three roots $\rho_+ = 0, 1/2$ or 1

$$0 = \frac{\lambda\rho_+(1-\rho_+)(1-2\rho_+)}{(\rho_+ + \lambda)(1-\rho_+ + \lambda)}$$

Completing the solution, we see that

- If $\rho_+ = 1$ then $\rho_{a+} = 1$.
- If $\rho_+ = 0$ then $\rho_{a-} = 1$.
- If $\rho_+ = 1/2$ then $\rho_{a+}/\rho_+ = 2\lambda/(1+2\lambda)$.

- Straightforward calculations show that the roots with $\rho_+ = 0$ or 1 are unstable while the one with $\rho_+ = 1/2$ is locally attracting.

- Three dimensional ODE: $(\rho_+, \rho_{a+}, \rho_{a-})$.

Voter model perturbations

- We want to use some techniques similar to those in Cox, Durrett, and Perkins to study the latent voter model when λ is large.
- At each site there is a rate λ Poisson process of “wake-up dots." For each neighbor y, x consults y at rate $1/d(x)$ where $d(x)$ is the degree of x.

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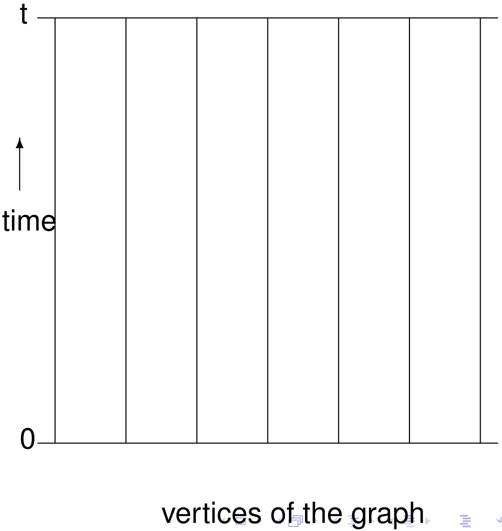
- At each site there is a rate λ Poisson process of "wake-up dots." For each neighbor y, x consults y at rate $1/d(x)$ where $d(x)$ is the degree of x.
- Number of consulting times for x between two wake-up dots is geometric with success probability $\lambda/(\lambda+1)$. The distribution is:

$$0 : \frac{\lambda}{1+\lambda} \quad 1 : \frac{\lambda}{(1+\lambda)^2} \quad 2 : \frac{\lambda}{(1+\lambda)^3} \quad \geq 3 : \frac{1}{(1+\lambda)^3}$$

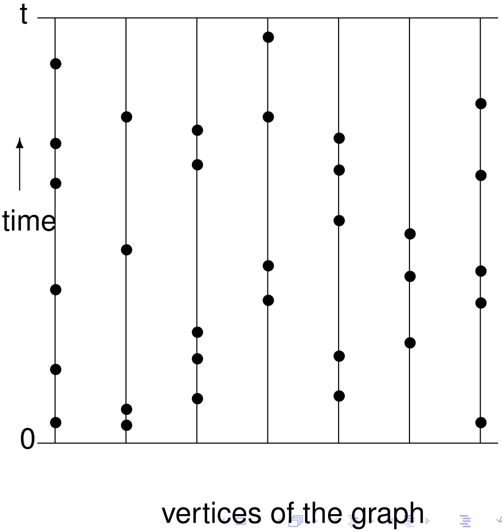
- Scale time at rate λ and in the limit we can ignore intervals with three or more arrows.

Graphical representation for rescaled latent voter model



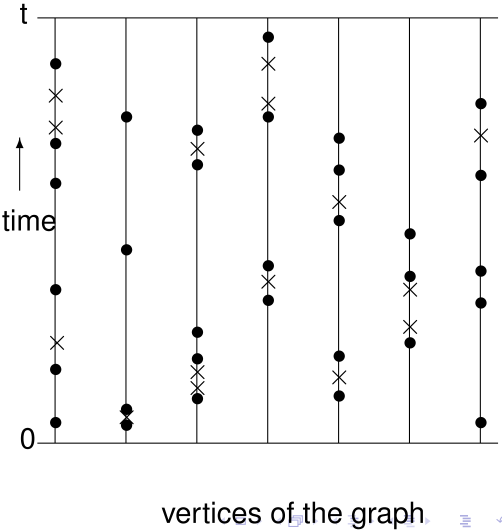
Graphical representation for rescaled latent voter model

- wake up dots at rate λ^2



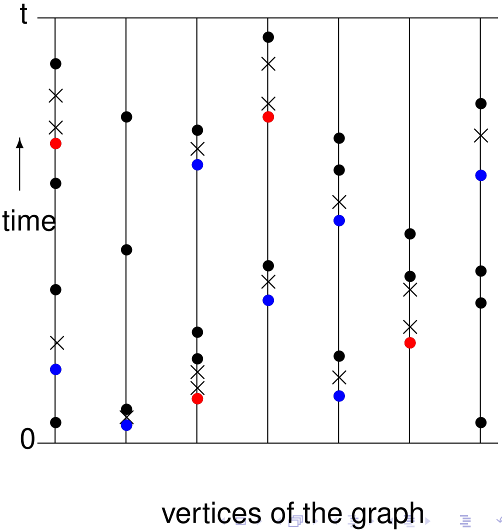
Graphical representation for rescaled latent voter model

- • wake up dots at rate λ^2
- $\times$ voting times at rate λ



Graphical representation for rescaled latent voter model

- ● wake up dots at rate λ^2
- $\times$ voting times at rate λ
- ● branching wake up dots at rate ≈ 1
- ● single wake up dots at rate λ



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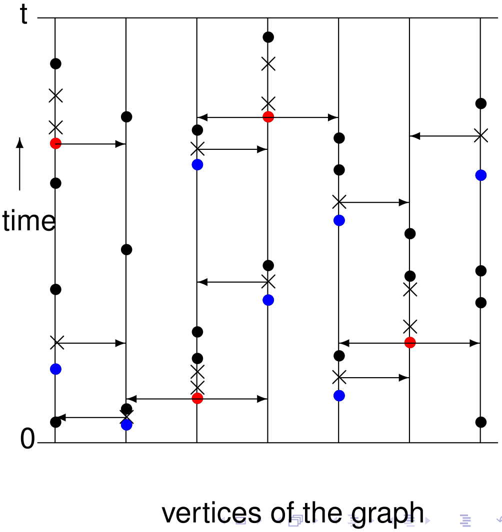
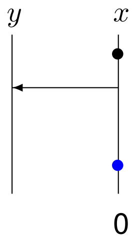


Changes in one interval

- If there is only one voting time in an interval between two wake-up dots then this is a voter event.



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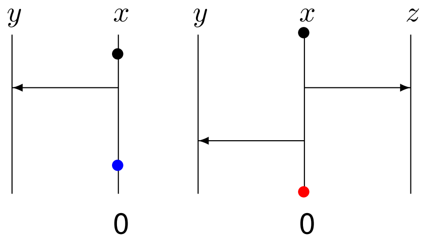


Considering the four cases we see that x will become 1 if and only at least one of x and y is a 1. (Redner's vacillating voter model.)

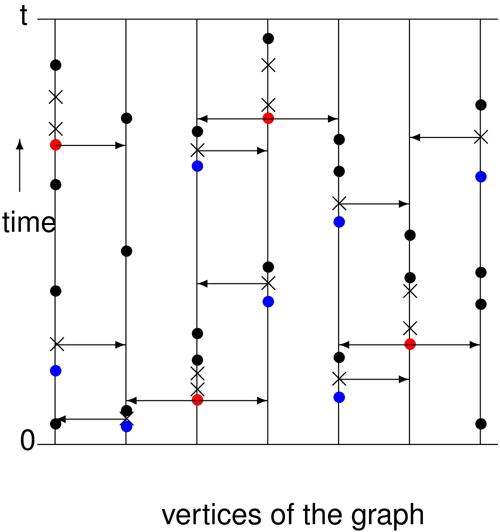
- If y is 1 then x flips to 1 but is inactive and ignores z. If y is 0 ...

Dual process

- To determine the state of x at time λt we work backwards.
- In an interval with one arrow $x \to y$ the particle follows the arrow and jumps, since x imitates y at that time.
- In an interval with two arrows, $x \to y$ and $x \to z$, x stays in the dual and we add y and z, since we need to know the state of x, y and z to see what will happen.

Dual process in the graphical representation

- $\bullet$ wake up dots at rate λ^2
- $\times$ voting times at rate λ
- red $\bullet$ branching wake up dots at rate ≈ 1
- blue $\bullet$ single wake up dots at rate $\approx \lambda$



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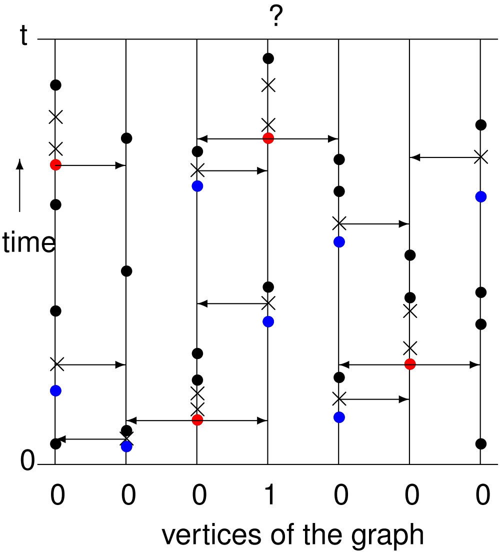

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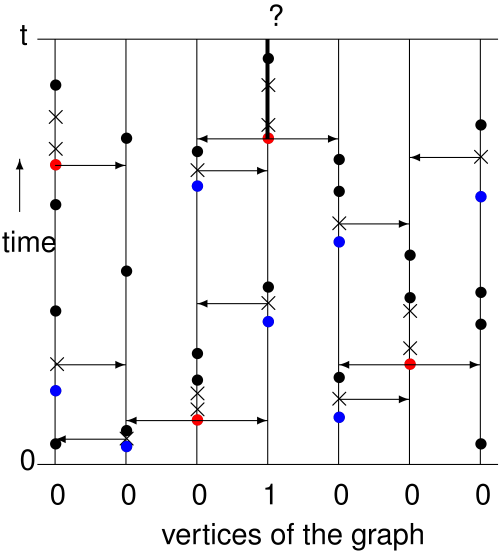

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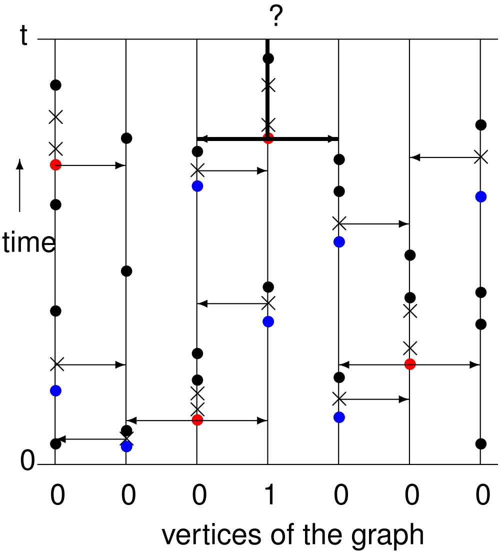

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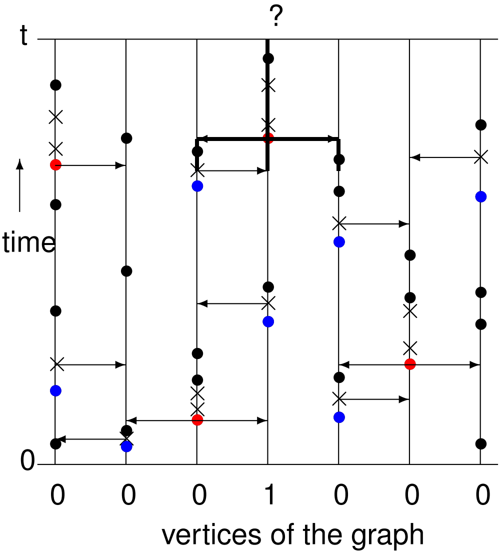

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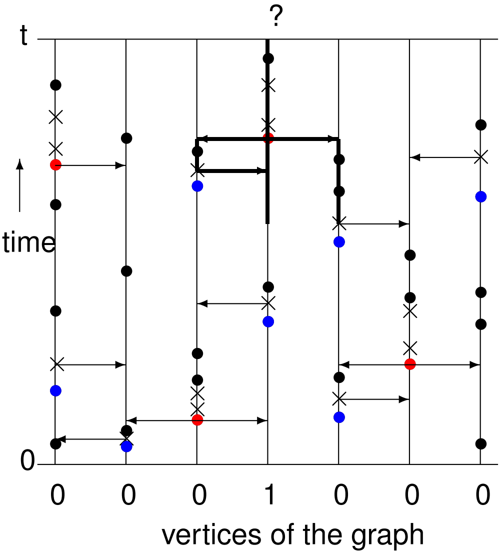

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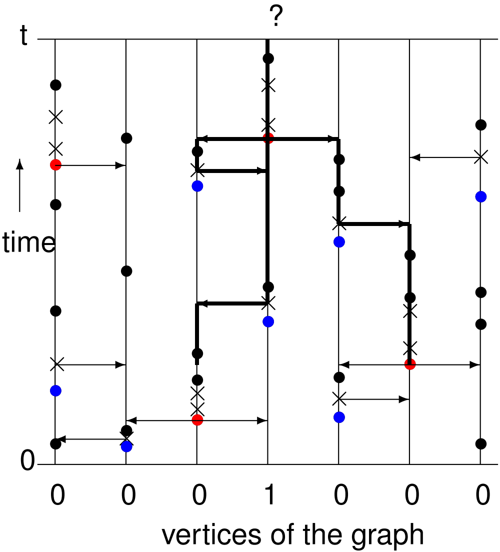

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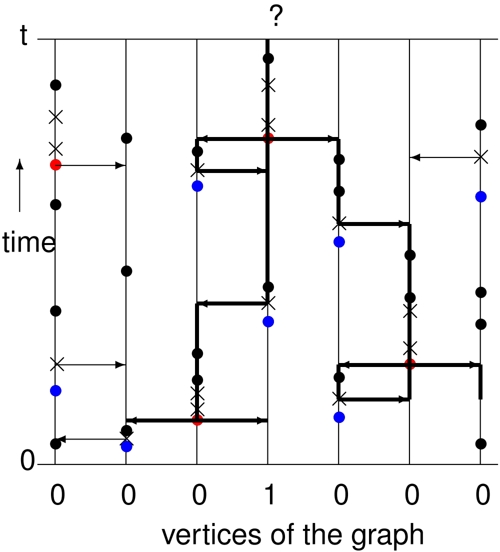

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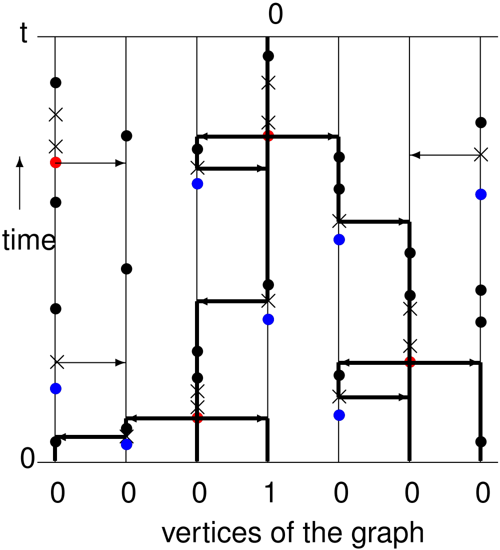
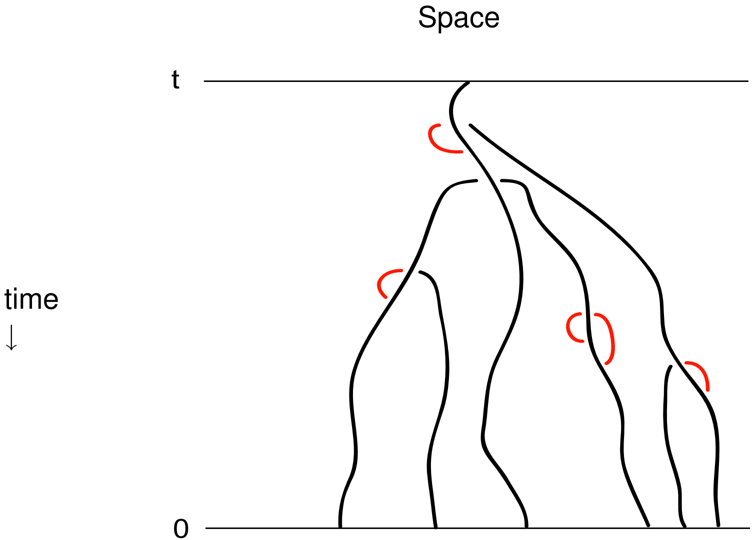


Schematic diagram of the dual



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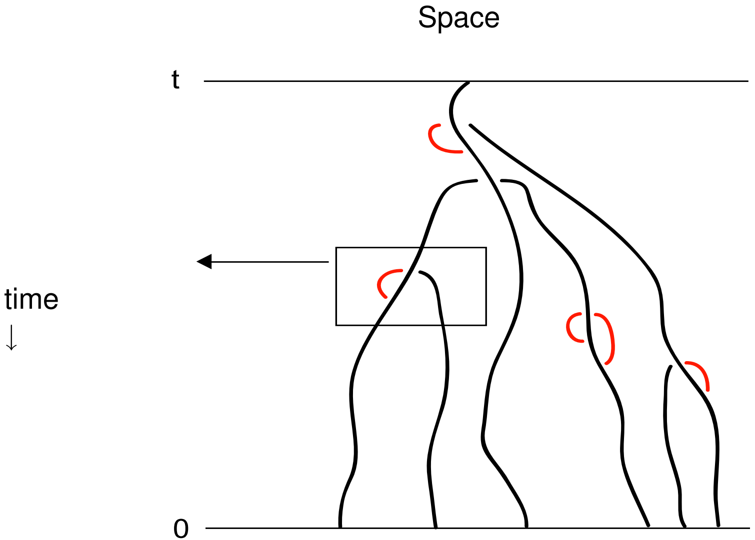

Schematic diagram of the dual

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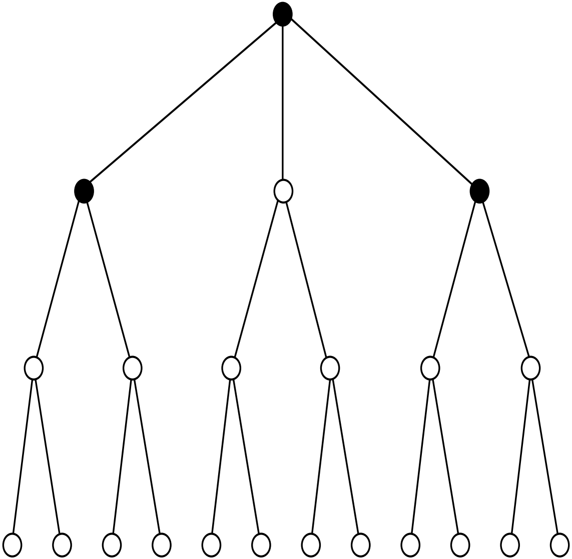

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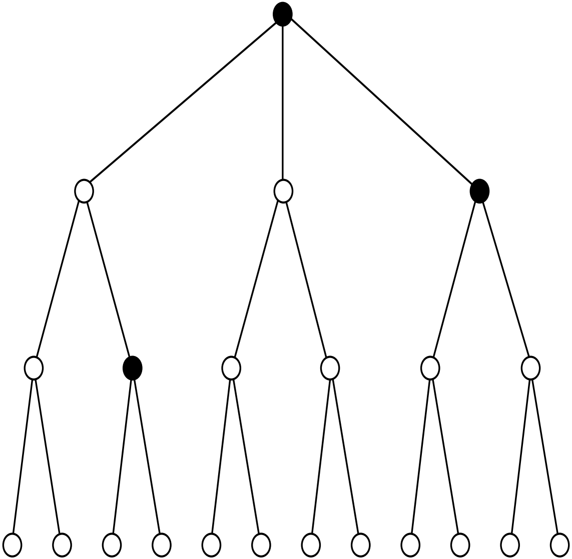

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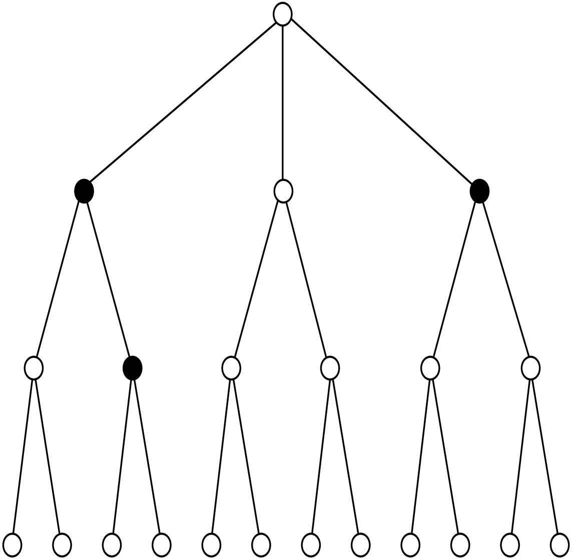

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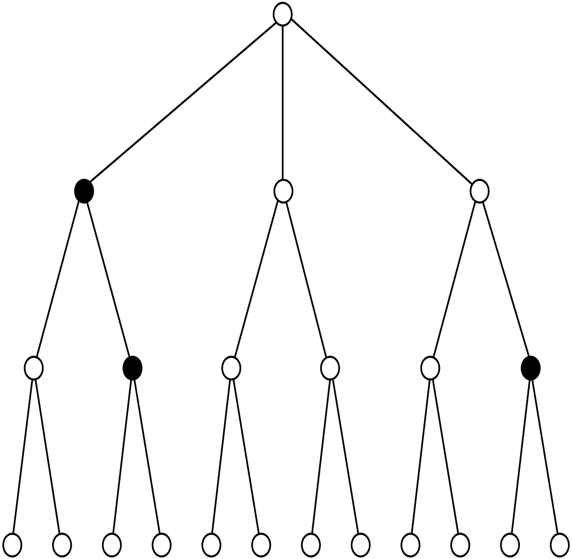

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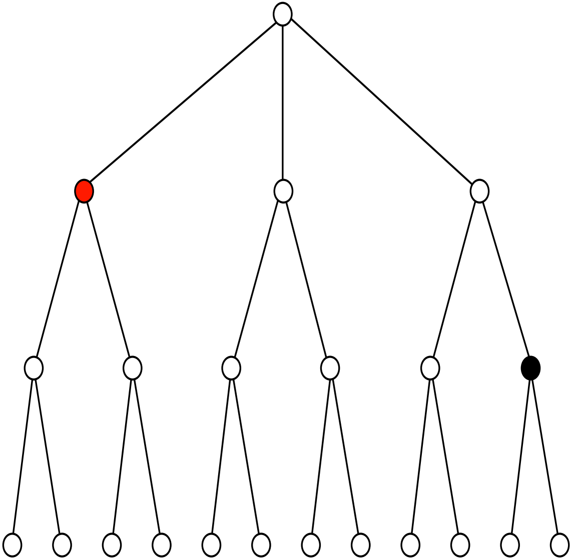

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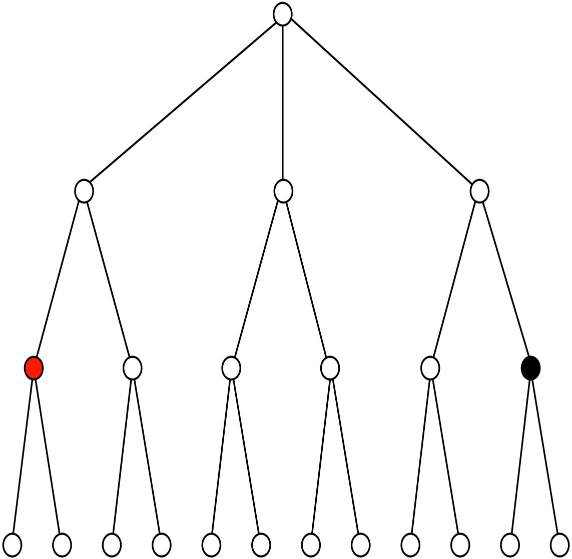

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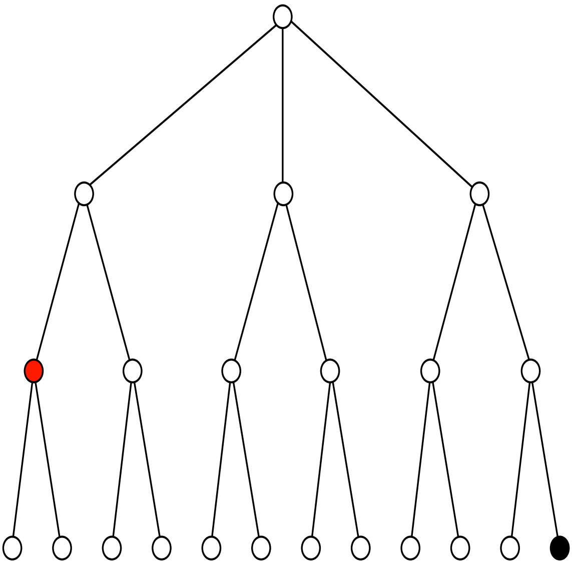

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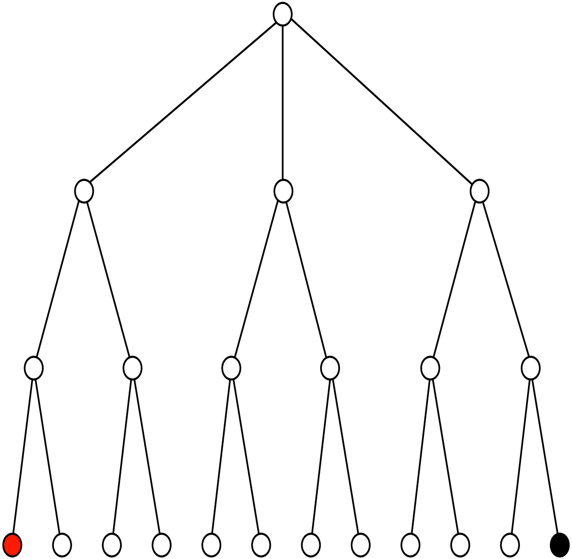

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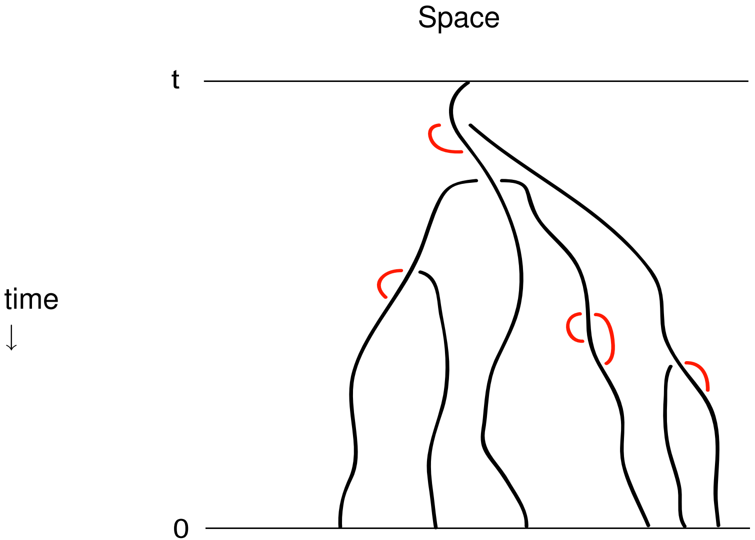

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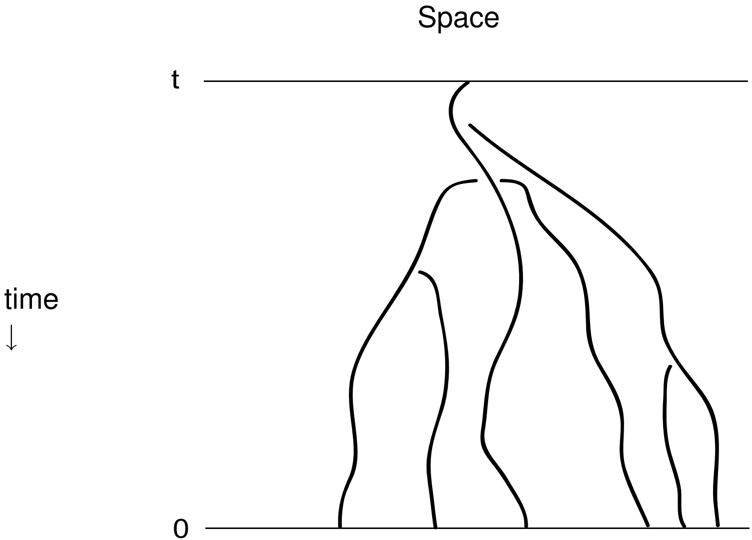

Limit theorem

For concreteness consider random regular graphs.

Theorem

If $\log n \ll \lambda \ll n/\log n$, *where* n *is the number of vertices of the random regular graph, then*

$$P(x \text{ has state 0 at time} \lambda t) \to \frac{1}{2} \text{ as } n \to \infty.$$

Similar result should be true for other locally tree-like random graphs.

Ingredients for the proof of limit theorem

- If the random walks from x, y, z all coalesce then we can ignore the event since nothing can happen in the branching random walk.
- If $xy|z$ or $xz|y$ i.e., two coalesce but avoid the other until time λt then the situation reduces to an ordinary voter arrow. This is also true if $x|yz$.

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- Let

$$r_n^1 := P(\text{ all three members of a family coalesce before } \epsilon \log n \text{ jumps})$$
$$r_n^2 := P(\text{none of the pairs in the family coalesce before } \epsilon \log n \text{ jumps})$$

- $r_n^1 \to r_1$ and $r_n^2 \to r_2$ as $n \to \infty$, e.g. the limit of r_2 is the probability of no coalescence among three random walks on the infinite tree with degree r starting from neighboring sites.

Sketch of the proof of the limit theorem

- The integral equation for $v_t = P(x$ has state 0 at time $\lambda t)$ is

$$v_t = \int_0^t (1-r_1)e^{-(1-r_1)s}\left[\frac{r_2}{1-r_1}\{v_{t-s}^3 + (1-v_{t-s})(1-(1-v_{t-s})^2)\}\right.$$
$$\left.+\frac{1-r_1-r_2}{1-r_1}v_{t-s}\right]\, ds + v_0 e^{-(1-r_1)t}.$$

- A little calculation shows that

$$v_t' = K v_t(1-v_t)(1-2v_t) \text{ for some constant } K = K(r_1, r_2)$$

Consensus time

Theorem

Consensus time is at least n^b for any $b < \infty$.

Reason

- The dual processes starting from distant vertices are asymptotically independent.
- Using Markov inequality for higher moments of the number of voters at time λt in state 0, it stays close to its mean with probability $\geq 1 - cn^{-b}$ for any $b < \infty$.

Why is the condition on λ required in the argument?

The conditions on λ guarantees that with high probability

- the particles become uniformly distributed on the graph between successive branching events,
- only those particles which are involved in the same branching coalesce,
- dual starting from distant particles are asymptotically independent.

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Future question

Q. What happens if λ is large but $O(1)$?

Thank you